Kinematic Modeling and Control Design for an Aerial Refueling Task

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This paper presents a kinematic simulation and control system design for the relative motion of two aircraft flying in close formation during automated aerial refueling. Up to now, linear control and approximated model based control is commonly used for that task with a maneuver defined in a tanker-fixed body frame. In the presented approach, exact Nonlinear Dynamic Inversion (NDI) is performed to design the relative position control of the receiver aircraft. The mathematical model used for control design is formulated in the NED frame of the receiver aircraft, leading to less complex equations.

The full kinematic equations of motion for the relative position dynamics are analyzed and derived with respect to three different reference frames. Due to the complexity of the relative motion, a suitable choice of reference frame in the mathematical formulation could simplify the design and implementation without sacrificing the performance. By the NDI transformation of the highly nonlinear relative dynamics of the aerial refueling motion, comparably simple linear control can be applied to fulfill the performance requirements of the aerial refueling task.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>B</td>
<td>Body-fixed frame</td>
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<tr>
<td>O</td>
<td>North-East-Down (NED) frame</td>
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<tr>
<td>O²</td>
<td>North-East-Down (NED) frame of the Receiver Aircraft</td>
</tr>
<tr>
<td>M_{O⁰}</td>
<td>Transformation matrix from O frame to B frame</td>
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<tr>
<td>(\omega_B)_B</td>
<td>[p q r]^T, angular rates around x, y and z axis of B frame</td>
</tr>
<tr>
<td>(\vec{r}<em>{12})</em>{O²}</td>
<td>relative position vector from point R₁ to point R₂, denoted in receiver NED frame O²</td>
</tr>
<tr>
<td>(\vec{v}<em>{12})</em>{O²}</td>
<td>relative velocity vector from point R₁ to point R₂, w.r.t. O² frame</td>
</tr>
<tr>
<td>(\vec{a}<em>{12})</em>{O²}</td>
<td>relative acceleration vector from point R₁ to point R₂, w.r.t. O² frame</td>
</tr>
<tr>
<td>(\vec{r}<em>{O²})</em>{O²}</td>
<td>absolute velocity vector of point R₂, w.r.t. O² frame</td>
</tr>
<tr>
<td>(\vec{a}<em>{O²})</em>{O²}</td>
<td>absolute acceleration vector of point R₂, w.r.t. O² frame</td>
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I. Introduction

In the past 20 years, as the visual navigation technology and data fusion technique advanced, aerial refueling operations have made great improvements in terms of accuracy and automation. There has been some recent work¹ ² dealing with the control system development and simulation for aerial refueling. Linear control and approximated model based control have been used and the dynamic equations were mostly formulated in the tanker body fixed frame. In this work, the focus is on simulation development and relative position control of the aerial refueling motion. The complete kinematic equations for the relative motion are explored in different frames seeking for the best performance and safety. The NDI controller is designed based on the derived kinematic models. In addition, the position error dynamics are rotated to the flight path frame so that the error feedbacks can have anisotropic properties in the positive and negative flight path direction. With the developed simulation environment,

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the probability of operation failure can be computed by Monte Carlo or subset simulation in the presence of modeling errors, navigation errors, and external disturbances.

The aerial refueling concept and simulation are introduced in Section II, including a preliminary maneuver definition with desired bounds. Section III gives the derivation of the relative kinematic equations, followed by the outer loop NDI control design. The last section gives the simulation results and conclusions.

II. Aerial Refueling Operation Simulation

A. Cruiser-Feeder Concept
The Cruiser-Feeder Concept is a pioneering idea for the air transport of the future, introduced in the European RECREATE project. The Cruiser continuously flies around the globe while the Feeder transports supplies and passengers from ground locations to the Cruiser and vice versa. Aerial refueling is one concept of Cruiser-Feeder operations. Generic four-engine large transport high fidelity aircraft simulation models are used for both cruiser and feeder, or receiver and tanker in the aerial refueling concept.

B. The Aerial Refueling Maneuver
The aerial refueling maneuver is defined and divided into several phases as shown in Fig. 1,
- Pre-Approach/Initiation Phase
- Approach Phase
- Astern Hold Phase
- Rendezvous/ Contact Phase
- Station Keeping Phase
- Departure Phase

![Figure 1. Maneuver phases](image-url)
According to the flight path definitions, the reference flight path can be generated in Matlab as shown in Fig. 2. The geometry of the flight path is defined with certain tolerances. The maneuver bounds are given in each phase, which are the top-level requirements of the complete system design.

**Figure 2. Aerial Refueling Maneuver**: green line is the reference relative flight path from initialization box to refueling position; magenta lines are the adequate maneuver bounds for the relative position control

**C. Reduced Simulation Model for Control Assessment**

The goal of this simulation environment is to identify the inner loop control performance. With the full 6 DOF high fidelity simulation model, it is not only very time consuming per simulation, but also difficult to quantify the inner loop performance in the presence of actuator dynamics, model uncertainties, and external disturbances. Hence, a reduced kinematic 6 DOF simulation model is constructed with a linear aerodynamic model and actuator dynamics. The primary goal of the reduced simulation model is to generate a realistic closed loop transfer function to represent the inner loop dynamics of the full model. First it should feature the most distinct aircraft dynamics; second, it can be evaluated with an automated process in Matlab Simulink; third, the simulation model should be able to provide the full state information for the outer loop controller design.

In the reduced model, the moment dynamics only capture the dynamics near the refueling states (altitude, velocity, weight, etc.). The following approximated rotational dynamic equations are used,

\[
\begin{align*}
\dot{\rho} &= -3.762\beta + 0.425r - 0.997p + M_\eta \eta \\
\dot{\vartheta} &= -1.678\alpha - 0.640q + L_\xi \xi \\
\dot{\varphi} &= 1.560\beta - 0.570r - 0.017p + N_\zeta \zeta
\end{align*}
\]

Engine dynamics and actuator dynamics of the control surface deflections are modeled with standard time lag elements of first and second order, as shown in Eq. (1) and (2).

\[
\delta_T(s) = \frac{1}{T_\tau s + 1} \delta_T
\]

\[
\eta(s) = \frac{\omega_\eta^2}{s^2 + 2\zeta_\eta \omega_\eta s + \omega_\eta^2} \eta(s)
\]

With the throttle commands and angular accelerations, the full states can be propagated using the standard aircraft rigid-body equations of motion. The translational state derivatives can be computed and propagated to the velocity and position. It is sufficient for the aerial refueling motion to capture only the linear region of the lift and side force curve, and to use a parabolic drag polar in the aerodynamic model.
Classical linear controllers are implemented for the inner loop. The control state variables are the load factor in z axis of kinematic frame $n_z$, bank angle $\Phi$ and thrust $T$. They can be calculated from the desired relative accelerations in the outer loop by inverting the translational dynamic equations, which will be derived in Section III.

The NDI control task aims on the relative position control outer loop as highlighted in the overall control structure in Fig. 3, the classical inner loop design is omitted here. Details can be found in the literature.\(^9,10\)

### III. Relative Dynamics Derivation and Simulation

The relative dynamics derivation starts from the relative position vectors as shown in Fig. 4. The tanker is denoted as 1, and the receiver is denoted as 2. The equations of the relative motion can be derived from differentiating the position vector equation until the acceleration vector of each aircraft appears in the equation. For derivation of the relative velocity and relative acceleration equation, the choice of the reference frame is of importance. There are several reasonable choices: 1. the receiver body-fixed frame $B_z$ as the final inner loop commands are given in the $B_z$ frame; 2. the tanker body-fixed frame $B_t$ as the tanker is in a steady state flight condition thus much more stable than the receiver. Also most of the previous work selects this frame as the reference; 3. The receiver NED frame $O_2$ which can be approximately considered as the inertial frame in the relative motion. The final equations of motion in each frame are given below. The detailed stepwise derivations are provided in the appendix.

#### A. Reference frame: Receiver body-fixed frame $B_z$

The relative position between the boom origin $H$ and the receptacle $F$ is,

$$\mathbf{r}_{BF} = \mathbf{r}_{HR} + \mathbf{r}_{BHR}$$

The relative velocity can be computed by differentiating the position equation in the same frame,

$$\mathbf{v}_{BF} = \mathbf{v}_{HR} + \mathbf{v}_{BHR}$$

The relative acceleration can be computed by differentiating the velocity vector,

$$\mathbf{a}_{BF} = \mathbf{a}_{HR} + \mathbf{a}_{BHR}$$

![Figure 3. Overall Control Implementation Structure](image)

![Figure 4. Relative position between boom origin $H$ and receptacle $F$](image)
B. Reference frame: Tanker body-fixed frame $B_1$

Similarly with the previous derivation, it starts with the position equation,
\[
(\dddot{\mathbf{p}}^{HF})_{B_1} = (\dddot{\mathbf{p}}^{HR_1})_{B_1} + (\dddot{\mathbf{p}}^{R_1R_2})_{B_1} + (\dddot{\mathbf{p}}^{R_2F})_{B_1}
\]

The relative velocity is,
\[
(\mathbf{v}^{HF})_{B_1} = (\ddot{\mathbf{v}}^{R_1R_2})_{B_1} + (\ddot{\mathbf{a}}^{R_1})_{B_1} \times (\ddot{\mathbf{p}}^{R_1F})_{B_1} + (\ddot{\mathbf{a}}^{R_2})_{B_1} \times (\ddot{\mathbf{p}}^{R_2F})_{B_1}
\]

The relative acceleration is,
\[
(\mathbf{a}^{HF})_{B_1} = (\dddot{\mathbf{v}}^{R_1R_2})_{B_1} + (\dddot{\mathbf{a}}^{R_1})_{B_1} \times (\ddot{\mathbf{p}}^{R_1F})_{B_1} + (\dddot{\mathbf{a}}^{R_2})_{B_1} \times (\ddot{\mathbf{p}}^{R_2F})_{B_1}
\]

C. Reference frame: Receiver NED frame $O_2$

The receiver NED frame can be used as inertia frame for the relative motion, as the rotations between $O_1$, $O_2$ and $E$ frames are negligible during the aerial refueling maneuver.

The relative position is,
\[
(\mathbf{P}^{HF})_{O_2} = (\mathbf{P}^{HR_1})_{O_2} + (\mathbf{P}^{R_1R_2})_{O_2} + (\mathbf{P}^{R_2F})_{O_2}
\]

The relative velocity is,
\[
(\mathbf{V}^{HF})_{O_2} = (\dot{\mathbf{V}}^{R_1R_2})_{O_2} + (\dot{\mathbf{a}}^{O_1B_1})_{O_1} \times (\mathbf{V}^{HR_1})_{O_1} + (\dot{\mathbf{a}}^{O_2B_2})_{O_2} \times (\mathbf{V}^{R_2F})_{O_2}
\]

The relative acceleration is,
\[
(\mathbf{A}^{HF})_{O_2} = (\dddot{\mathbf{V}}^{R_1R_2})_{O_2} + (\dot{\mathbf{v}}^{O_1B_1})_{O_1} \times (\ddot{\mathbf{V}}^{HR_1})_{O_1} + (\dot{\mathbf{v}}^{O_2B_2})_{O_2} \times (\ddot{\mathbf{V}}^{R_2F})_{O_2}
\]

The kinematic equations above are much simpler compared to the other cases because the cross coupling terms due to frame rotations vanish. A reduced model propagating the relative acceleration to the velocity and position was used to verify the validity of the relative acceleration equations.

IV. NDI Control Design for the Relative Motion

In this section, a control system based on nonlinear dynamic inversion\(^{11, 12}\) is designed for the relative motion. The dynamic inversion implementation normally consists of three parts: a reference model, an error controller, and the dynamic inversion. The reference model generates smooth trajectories which are physically feasible for the plant and provide the higher order derivatives of the reference signals to the feedforward control. The dynamic inversion part inverts the dynamics of the nominal plant and transforms the nonlinear plant to an equivalent linear one so that a linear error controller can be designed. The error controller mainly accounts for the inversion error and external disturbances.

In this case, the reference signals are given by the trajectory generation according to the maneuver definition. The dynamic inversion and the error controller are described below.

A. Error Controller

As there are two integrator levels between the system output $\mathbf{y}^{HF}_{O_2}$ and the inner loop commands, it is a second order system for the outer loop control. A typical error controller for this second order system is a linear PD controller. The pseudo control $\mathbf{v}$ is the desired relative acceleration $(\mathbf{a}^{HF})_{O_2}$ resulting from the linear control law given below,
\[
\mathbf{v} = (\mathbf{a}^{HF})_{O_2} = (\dddot{\mathbf{P}}^{HF})_{O_2} + K_d ((\dddot{\mathbf{P}}^{HR_1})_{O_2} - (\dddot{\mathbf{P}}^{R_1R_2})_{O_2}) + K_p ((\dddot{\mathbf{P}}^{HF})_{O_2} - (\dddot{\mathbf{P}}^{HR_1})_{O_2} - (\dddot{\mathbf{P}}^{R_1R_2})_{O_2})
\]

The denotation ‘des’ means it is a desired value and ‘r’ means it is a reference signal from the flight path generation. For a perfect model without actuator dynamics and uncertainties, the following error dynamics can be obtained,
\[
(\dddot{\mathbf{P}}^{HF})_{O_2} = (\dddot{\mathbf{P}}^{HR_1})_{O_2} + K_d ((\dddot{\mathbf{P}}^{HR_1})_{O_2} - (\dddot{\mathbf{P}}^{R_1R_2})_{O_2}) + K_p ((\dddot{\mathbf{P}}^{HF})_{O_2} - (\dddot{\mathbf{P}}^{HR_1})_{O_2} - (\dddot{\mathbf{P}}^{R_1R_2})_{O_2}) = 0
\]
In Laplace domain, and define \( \ddot{\mathbf{e}} = (\mathbf{g}^{HF})_{O_2,F} - (\mathbf{g}^{HF})_{O_2} \):

\[
\ddot{\mathbf{e}} + K_p \dot{\mathbf{e}} + K_e \mathbf{e} = 0
\]

\[
(s^2 + K_0s + K_p)\ddot{\mathbf{e}} = 0
\]

B. Nonlinear Dynamic Inversion of the Relative Kinematics

After the error control, the pseudo control \((\mathbf{a}^{HF})_{O_2,des}\) is obtained. The dynamic inversion module computes the inner loop commands by inverting the dynamic equations from the pseudo control. As shown in Fig. 5, the pseudo control signal is first inverted by the relative dynamic equation, from the desired relative acceleration \((\mathbf{a}^{HF})_{O_2,des}\) to the desired absolute acceleration \((\mathbf{a}^{RF})_{O_2}\). In the receiver aircraft dynamic inversion, the later on defined load factor vector \((\mathbf{n}^{R})_{K_2,des}\) can be computed from the desired absolute acceleration and eventually the inner loop commands can be computed from the load factor vector.

First, by inverting the relative dynamics given in Eq. (5), the receiver absolute acceleration in NED frame can be obtained,

\[
(\mathbf{a}^{R})_{O_2} = (\mathbf{a}^{HF})_{O_2} + (\mathbf{a}^{RI})_{O_2} - (\mathbf{a}^{O})_{O_2} - (\mathbf{a}^{R})_{O_2} - (\mathbf{a}^{R})_{O_2} - (\mathbf{a}^{R})_{O_2}
\]

\[
= \mathbf{a}^{RI} \times (\mathbf{a}^{RF})_{O_2}
\]

The \((\mathbf{n}^{R})_{O_2}\) can be expressed by the load factor vector in receiver aircraft kinematic frame \(K_2\), and \(M_{O_2K_2}\) is the transformation matrix from \(K_2\) frame to \(O_2\) frame.

\[
(\mathbf{n}^{R})_{O_2} = M_{O_2K_2} (\mathbf{n}^{R})_{K_2} + (\mathbf{g}^{R})_{O_2}
\]

where \((\mathbf{g}^{R})_{O_2}\) is the gravitational vector,

\[
(\mathbf{g})_{O_2} = \begin{bmatrix} 0 \\ 0 \\ 9.8 \end{bmatrix}
\]

\((\mathbf{R})_{K_2}\) is the specific force vector in kinematic frame, i.e. the non-gravitational force normalized by mass and the load factor vector is defined by the specific force vector normalized by the gravitational constant \(g\).

\[
(\mathbf{n}^{R})_{K_2} = \frac{(\mathbf{n}^{R})_{O_2}}{g}
\]

Hence, the inversion equation for \((\mathbf{n}^{R})_{K_2}\) can be derived,

\[
(\mathbf{n}^{R})_{K_2} = \frac{M_{K_2O_2}}{g} (\mathbf{n}^{R})_{O_2} - (\mathbf{g})_{O_2}
\]

Then the load factor in the kinematic frame can be allocated to the inner loop commands as follows,

\[
\{T_c \approx mg \cdot n_x, \quad \Phi_c \approx \tan n_y, \quad n_{z_k} \approx n_z\}
\]

That completes the dynamic inversion from the pseudo control to the inner loop commands. The outer loop control structure is shown in Fig. 5, which completes the overall control structure given in Fig. 3.
V. Simulation Results and Conclusion

The overall flight path from initialization until the stationary keeping phase are simulated in 400s. Before the simulation, all components have been tested and verified separately. Actuator dynamics and external disturbances with moderate intensity level have been considered here. The inner loop controller is tuned to mimic the conventional behavior of the given aircraft model, i.e. relatively slow inner loop dynamics are featured. The time constant of the thrust dynamics is approximated as 2 s, and the control surface deflections have a time constant of 0.25 s.

In the following figures the simulated path tracking performance is presented. In Fig. 6 and Fig. 7, by comparison of the reference commands and the state signals, the tracking performance of the relative position and velocity are shown. Without any disturbances, the tracking performance is excellent. Fig. 8 and Fig. 9 show the tracking performance with moderate turbulences. It can be seen that under the moderate turbulences, the tracking performance are still within the adequate bounds. The relative position tracking errors are shown in Fig. 10; and the path geometry in 2D plot is shown in Fig. 11, with a close up view of the astern hold and stationary keeping phase. The outer loop control design has shown promising performance. Relative position tracking errors are rather small and within the required bounds. The simulation framework for further analysis has been established.

In future work, the kinematic simulation environment can be used to identify the required inner loop control performance of the receiver to ensure robust and safe aerial refueling not only with perfect state feedback but also under the presence of sensor measurements errors and plant modeling uncertainties. In addition, a parameter sensitivity analysis can be performed with the developed simulation environment; especially the effect of the signals and parameters used in the relative dynamic inversion can be analyzed under biased noises and time delays.

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This publication reflects only the authors' views. The European Union is not liable for any use that may be made of the information contained therein.

References

8. RECREATE project, URL: http://www.cruiser-feeder.eu [cited 5th June 2013]
Figure 6. Relative Position Tracking without Turbulences

Figure 7. Relative Velocity Tracking without Turbulences
Figure 8. Relative Position Tracking with Moderate Turbulences

Figure 9. Relative Velocity Tracking with Moderate Turbulences
Figure 10. Position Tracking Error with Moderate Turbulences

Figure 11. Path Control Performance
(red * points are (from left to right) initialization, astern hold and stationary keeping)
A. Reference frame: Receiver body-fixed frame $B_2$

The relative position between the boom origin H and the receptacle F is,

$$(\mathbf{r}^{HF})_{B_2} = (\mathbf{r}^{HR})_{B_2} + (\mathbf{r}^{R,F})_{B_2}.$$  

The relative velocity can be computed by differentiating the position equation in the same frame,

$$(\mathbf{v}^{HF})_{B_2} = \frac{d}{dt}(\mathbf{r}^{HF})_{B_2} = (\mathbf{v}^{HR})_{B_2} + (\mathbf{v}^{R,F})_{B_2}.$$  

The relative acceleration can be computed by differentiating the velocity vector,

$$(\mathbf{a}^{HF})_{B_2} = \frac{d}{dt}(\mathbf{v}^{HF})_{B_2} = (\mathbf{a}^{R,F})_{B_2} + (\mathbf{a}^{HR})_{B_2} + (\mathbf{a}^{B,E})_{B_2}.$$  

Because of the rigid body constraint with $(\mathbf{r}^{HR})_{B_2} = 0$ and $(\dot{\mathbf{r}}^{R,F})_{B_2} = 0$ it can be written as:

$$(\mathbf{v}^{HF})_{B_2} = (\dot{\mathbf{r}}^{R,F})_{B_2} + (\mathbf{a}^{B,E})_{B_2} \times (\dot{\mathbf{r}}^{HR})_{B_2}.$$  

To simplify the equation the relative rotation $(\mathbf{q}^{B_2B_1})$ can be factorized.

$$(\mathbf{v}^{HF})_{B_2} = (\dot{\mathbf{r}}^{R,F})_{B_2} + (\mathbf{a}^{B,E})_{B_2} \times (\dot{\mathbf{r}}^{HR})_{B_2}.$$  

The relative acceleration can be computed by differentiating the velocity vector,

$$(\mathbf{a}^{HF})_{B_2} = \frac{d}{dt}(\mathbf{v}^{HF})_{B_2} = (\mathbf{a}^{R,F})_{B_2} + (\mathbf{a}^{HR})_{B_2} + (\mathbf{a}^{B,E})_{B_2} \times (\dot{\mathbf{r}}^{HR})_{B_2}.$$  

Note that:

$$(\mathbf{a} \times \mathbf{c}) \times \mathbf{b} + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\mathbf{c} \times \mathbf{b})$$  

Hence,

$$(\mathbf{a}^{HR})_{B_2} = (\dot{\mathbf{r}}^{R,F})_{B_2} + (\mathbf{a}^{B,E})_{B_2} \times (\dot{\mathbf{r}}^{HR})_{B_2}.$$  

Convert to the receiver acceleration w.r.t. the inertia frame $(\mathbf{a}^{R,F})^{EE}_{B_2},$

$$(\mathbf{a}^{R,F})^{EE}_{B_2} = (\mathbf{a}^{R,F})^{EE}_{B_2} + (\mathbf{a}^{B,E})^{EE}_{B_2} \times (\dot{\mathbf{r}}^{HR})_{B_2}.$$  

Hence, the final relative acceleration equation derived in receiver body-fixed frame is,

$$(\mathbf{a}^{HF})_{B_2} = (\mathbf{a}^{R,F})^{EE}_{B_2} + (\mathbf{a}^{B,E})^{EE}_{B_2} \times (\dot{\mathbf{r}}^{HR})_{B_2}.$$
B. Reference frame: Tanker body-fixed frame $B_1$

The derivation starts with the relative position equation,
$$(\mathbf{r}^{HF})_{B_1} = (\mathbf{r}^{HR})_{B_1} + (\mathbf{r}^{R_2})_{B_1} + (\mathbf{r}^{PF})_{B_1}$$

The relative velocity can be obtained by differentiating the position,
$$(\mathbf{V}^{HF})_{B_1} = \frac{d}{dt} (\mathbf{r}^{HF})_{B_1} = (\mathbf{V}^{HR})_{B_1} + (\mathbf{V}^{R_2})_{B_1} + (\mathbf{V}^{PF})_{B_1}$$

Further differentiating the above equation, we can obtain the relative acceleration,
$$(\mathbf{a}^{HF})_{B_1} = \frac{d}{dt} (\mathbf{V}^{HF})_{B_1} = (\mathbf{a}^{HR})_{B_1} + (\mathbf{a}^{R_2})_{B_1} + (\mathbf{a}^{PF})_{B_1}$$

Because of the rigid body constraint with $$(\mathbf{r}^{HR})_{B_1} = 0$$ and $$(\mathbf{r}^{PF})_{B_1} = 0$$ it can be written as:
$$(\mathbf{V}^{HF})_{B_1} = (\mathbf{r}^{R_2})_{B_1} + (\mathbf{V}^{R_2})_{B_1} + (\mathbf{a}^{R_2})_{B_1}$$

To simplify the equation the relative rotation $$(\mathbf{a}^{R_1})_{B_1}$$ can be factorized.

Further differentiating the above equation, we can obtain the relative acceleration,
$$(\mathbf{a}^{HF})_{B_1} = (\mathbf{a}^{R_1})_{B_1} + (\mathbf{a}^{R_2})_{B_1}$$

Note: $$(\mathbf{a} \times \mathbf{c}) \times \mathbf{b} + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\mathbf{c} \times \mathbf{b})$$

Hence,
$$(\mathbf{a}^{HF})_{B_1} = (\mathbf{a}^{R_1})_{B_1} + (\mathbf{a}^{R_2})_{B_1}$$

Note: $$(\mathbf{a} \times \mathbf{c}) \times \mathbf{b} + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\mathbf{c} \times \mathbf{b})$$

Hence,
$$(\mathbf{a}^{HF})_{B_1} = (\mathbf{a}^{R_1})_{B_1} + (\mathbf{a}^{R_2})_{B_1}$$
C. Reference frame: Receiver NED frame $O_2$

The relative position is,

\[
(\hat{r}^{HF})_{O_2} = (\hat{r}^{HR})_{O_2} + (\hat{r}^{R_F})_{O_2} + (\hat{r}^{R_2})_{O_2}
\]

The relative velocity,

\[
(\hat{v}^{HF})_{O_2} = \frac{d}{dt}(\hat{r}^{HF})_{O_2} = (\hat{r}^{HR})_{O_2} + (\hat{r}^{R_F})_{O_2} + (\hat{r}^{R_2})_{O_2}
\]

\[
(\hat{v}^{HF})_{O_2} = (\hat{r}^{HR})_{O_2} + (\hat{\omega}^{R_2B})_{O_2} \times (\hat{r}^{HR})_{O_2} + (\hat{r}^{R_F})_{O_2} + (\hat{\omega}^{R_2B})_{O_2} \times (\hat{r}^{R_F})_{O_2}
\]

Because of the rigid body constraint with $(\hat{r}^{HR})_{O_2} = 0$ and $(\hat{r}^{R_2F})_{O_2} = 0$ it can be written as:

\[
(\hat{v}^{HF})_{O_2} = (\hat{r}^{R_F})_{O_2} + (\hat{\omega}^{R_2B})_{O_2} \times (\hat{r}^{R_F})_{O_2}
\]

The relative acceleration,

\[
(\hat{a}^{HF})_{O_2} = \left(\frac{d}{dt}\right)^2(\hat{r}^{HF})_{O_2} = (\hat{r}^{HR})_{O_2} + (\hat{r}^{R_F})_{O_2}
\]

\[
(\hat{a}^{HF})_{O_2} = (\hat{r}^{HR})_{O_2} + (\hat{\omega}^{R_2B})_{O_2} \times (\hat{r}^{HR})_{O_2} + (\hat{r}^{R_F})_{O_2} \times (\hat{r}^{HR})_{O_2}
\]

Again apply the rigid body constraint, $(\hat{r}^{HR})_{O_2} = 0$ and $(\hat{r}^{R_2F})_{O_2} = 0$ and note the cross product of the same vector is zero, e.g., $(\hat{\omega}^{R_2B})_{O_2} \times (\hat{\omega}^{R_2B})_{O_2} = 0$.

\[
(\hat{a}^{HF})_{O_2} = (\hat{r}^{R_F})_{O_2} + (\hat{\omega}^{R_2B})_{O_2} \times (\hat{r}^{R_F})_{O_2}
\]

Ignoring the rotation rates, $\hat{\omega}^{R_2B}_{O_2} = 0$, and $\hat{\omega}^{E2}_{O_2} = 0$. The NED frames cannot be assumed to be the same for the absolute position vector, but the absolute velocity and acceleration vectors are valid to make the $O_1 = O_2$ assumption.

\[
(\hat{v}^{HF})_{O_2} = (\hat{v}^{R_F})_{O_2} + (\hat{\omega}^{R_2B})_{O_2} \times (\hat{v}^{R_F})_{O_2}
\]

And the acceleration as well,

\[
(\hat{a}^{HF})_{O_2} = (\hat{a}^{R_F})_{O_2} + (\hat{\omega}^{R_2B})_{O_2} \times (\hat{v}^{R_F})_{O_2}
\]

Where, the relative velocity can be expressed as,

\[
(\hat{v}^{HR})_{O_2} = (\hat{r}^{R_2})_{O_2} - (\hat{r}^{R_1})_{O_2} = (\hat{r}^{R_2})_{O_2} - M_{O_2E} \hat{v}^{R_1}_{E}
\]

\[
(\hat{v}^{HF})_{O_2} = (\hat{v}^{HR})_{O_2} + (\hat{r}^{R_F})_{O_2} + (\hat{\omega}^{R_2B})_{O_2} \times (\hat{r}^{R_F})_{O_2}
\]

Ignoring the rotations between NED and ECEF frames,

\[
(\hat{v}^{R_2})_{O_2} = (\hat{v}^{R_2})_{E} \approx (\hat{v}^{R_2})_{O_2} - (\hat{r}^{R_1})_{O_1}
\]

\[
(\hat{v}^{HF})_{O_2} = (\hat{v}^{R_2})_{E} - (\hat{v}^{R_1})_{O_1} + (\hat{\omega}^{R_2B})_{O_2} \times (\hat{r}^{HR})_{O_2} + (\hat{r}^{R_F})_{O_2}
\]

The relative acceleration,
\[(\ddot{\mathbf{a}}_{R_{1}})^{O_{2}}_{O_{1}} \approx (\ddot{\mathbf{a}}_{R_{2}})^{E_{2}}_{O_{2}} - (\ddot{\mathbf{a}}_{R_{1}})^{E_{1}}_{O_{2}}\]
\[(\ddot{\mathbf{a}}_{R_{2}})^{O_{2}}_{O_{1}} \approx (\ddot{\mathbf{a}}_{R_{2}})^{E_{2}}_{O_{2}} + (\omega^{O_{1}}_{R_{2}})^{B_{1}}_{O_{1}} \times (\mathbf{H}_{R_{1}})^{B_{1}}_{O_{1}} + (\omega^{O_{2}}_{R_{2}})^{B_{2}}_{O_{2}} \times (\mathbf{H}_{R_{2}})^{B_{2}}_{O_{2}}
+ (\omega^{O_{1}}_{R_{2}})^{B_{1}}_{O_{1}} \times [(\omega^{O_{1}}_{R_{2}})^{B_{1}}_{O_{1}} \times (\mathbf{H}_{R_{1}})^{B_{1}}_{O_{1}}] + (\omega^{O_{2}}_{R_{2}})^{B_{2}}_{O_{2}} \times [(\omega^{O_{2}}_{R_{2}})^{B_{2}}_{O_{2}} \times (\mathbf{H}_{R_{2}})^{B_{2}}_{O_{2}}]\]

D. System of Notation:

- **Matrix:** Capital letter, **bold** and non-italic
- **Vector:** small letter, **bold** and non-italic
- **Scalar:** small letter, non-bold and *italic*

For any Euclidean vector \(\mathbf{x}\), the following declaration scheme is applied:
\[(\mathbf{x})^{Reference\;point}_{Reference\;Frame}^{Reference\;Frame}_{Reference\;Frame}^{Reference\;Frame}_{Reference\;Frame}^{Reference\;Frame}\]

For example, the velocity derivative \((\dot{\mathbf{v}}^{G}_{K})^{E_{1}}_{K}\) means,

- **Referenced to point G**
- **A velocity relative to the E-frame has been differentiated with respect to the K-frame**
- **K-frame is Notation Frame**
- **Type of Acceleration addressed: K-kinematic, W-wind, A-Aerodynamic**